

# Oscillator glass in the generalized Kuramoto model: synchronous disorder and two-step relaxation

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We consider a very general form of the Kuramoto model (KM) and derive equations for the macroscopic parameters of its stationary states. Surprisingly, we discover that a class of simple, analytically treatable KM variants can exhibit the long-sought oscillator glass behavior, which represents a kind of synchronous disorder. We derive exact conditions for the appearance of the glassy states and study their exotic properties. Our findings offer the possibility of creating such states in real systems described by the KM, which may lead to a diversity of interesting phenomena.

The Kuramoto model (KM) [1] was developed to provide an analytically tractable description of the large populations of coupled oscillators that appear in many branches of science [2, 3]. It was used e.g. to describe the collective behavior of lasers [4, 5], neurons in the brain [6], Josephson junction circuits [7], even humans [8]. We consider the KM in its very general form

$$\dot{\theta}_i = \omega_i - \frac{k_i}{N} \sum_{j=1}^N q_j \sin(\theta_i - \theta_j - \beta_i - \gamma_j), \quad (1)$$

where  $\theta_i(t)$  and  $\omega_i$  are respectively the  $i$ th oscillator's phase and natural frequency,  $N$  is their number,  $k_i q_j$  ( $\beta_i + \gamma_j$ ) represent the coupling strengths (phase lags) between the  $i$ th and  $j$ th oscillators, and parameters  $\Gamma_i \equiv \{\omega_i, k_i, q_i, \beta_i, \gamma_i\}$  are drawn from a joint probability density  $G(\Gamma) \equiv G(\omega, k, q, \beta, \gamma)$ . The case considered earlier [9] corresponds to (1) with  $q_i = 1, \beta_i = \gamma_i = 0$ .

The KM has not previously been treated in such a general form (1), although it includes the many KM modifications and extensions studied earlier as particular cases. Thus, not much is known about possible behavior that it may exhibit. Daido reported evidence [10] that the KM with random  $k_i q_j$  and  $\beta_i + \gamma_j$  can undergo a glass transition but, due to the complexity of the model, the resultant ‘‘oscillator glass’’ and its properties still remained mysterious [10–14]. To illuminate this issue, some authors (e.g. [15]) attempted to find the same state in simpler models, but were not successful in this respect.

In this Letter, we report our discovery of the oscillator glass in a class of simple, analytically treatable, models included within (1). We first generalize the framework of [9] to encompass (1), and then proceed to consideration of the glassy states, deriving conditions for their appearance and studying their properties.

*Background.* The oscillators' collective behavior in (1) can be described by two complex parameters

$$Z \equiv R e^{i\Psi} \equiv \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}, \quad Y \equiv W e^{i\Phi} \equiv \frac{1}{N} \sum_{j=1}^N q_j e^{i\gamma_j} e^{i\theta_j}, \quad (2)$$

where  $Z$  is the mean field whose amplitude  $R$  quantifies the degree of agreement between the oscillators' phases  $\theta_i$ , while  $Y$  represents the weighted mean field; when

$q_i = 1, \gamma_i = 0$ , the fields coincide. The KM (1) can also be rewritten as  $\dot{\theta}_i = \omega_i - k_i W \sum_j \sin(\theta_i - \theta_j - \beta_i - \Phi)$ .

In the continuum limit  $N \rightarrow \infty$ , we drop indices and treat (1) using the probability density function (PDF)  $f(\theta, \Gamma, t)$ , representing the probability that the oscillator has parameters  $\Gamma$  and phase  $\theta$  at time  $t$ . The PDF can usually [16–19] be represented by the OA ansatz [20]

$$f(\theta, \omega, K, t) = \frac{G(\Gamma)}{2\pi} \left[ 1 + 2\text{Re} \frac{\alpha(\Gamma, t) e^{i\theta}}{1 - \alpha(\Gamma, t) e^{i\theta}} \right]. \quad (3)$$

Following the conventional procedure [20], one can show that in our case  $\alpha$  should satisfy

$$\frac{\partial \alpha}{\partial t} + i\omega\alpha + \frac{kW}{2} (\alpha^2 e^{i(\beta+\Phi)} - e^{-i(\beta+\Phi)}) = 0, \quad (4)$$

$$Z = R e^{i\Psi} = \int \alpha^*(\Gamma, t) G(\Gamma) d\Gamma, \quad (5)$$

$$Y = W e^{i\Phi} = \int q e^{i\gamma} \alpha^*(\Gamma, t) G(\Gamma) d\Gamma, \quad (6)$$

where  $d\Gamma \equiv dk dq d\beta d\gamma$  and integration is taken over the whole domain if unspecified  $[(-\infty, \infty)$  for  $k, q$  and  $[-\pi, \pi]$  for  $\beta, \gamma$ ]. Note that the difference  $\Phi - \Psi$  represents another meaningful parameter: it is invariant under  $\theta \rightarrow \theta - \theta_0(t)$ , unlike  $\Phi$  and  $\Psi$  individually.

*Parameter normalization.* The model (1) is invariant under both  $k_i \rightarrow -k_i, \beta_i \rightarrow \beta_i + \pi$  and rescalings  $k_i \rightarrow k_i/a, q_i \rightarrow a q_i$ . To avoid ambiguity, one can fix the normalization of  $q_i$  and specify a rule for choosing the sign of  $k_i$ ; but different normalizations might be preferred in different situations [21]. Because the normalization can be fixed at any time, we will leave this ambiguity to preserve generality. Note, that  $W$  (but not  $R$ ) changes under the  $(k, q)$ -rescaling and thus can be higher than unity, but that  $|k|W$  is invariant.

*Terminology and notation.* We adopt similar terminology to [9, 22]. The KM form (1) is invariant under  $\tilde{\theta} = \theta - \Omega t$ , which just changes the distribution  $G(\omega, k, q, \beta, \gamma) \rightarrow G(\omega + \Omega, k, q, \beta, \gamma)$ , with  $\omega$  always denoting the natural frequency in the current frame (with phases  $\tilde{\theta}$ ). Thus, one can consider the KM in different frames rotating at frequency  $\Omega$  with respect to some fixed frame. For the latter we select the frame with zero mean frequency  $\langle \omega \rangle \equiv \int \omega G(\Gamma) d\Gamma = 0$  and call it the *natural*

frame; the distribution  $G(\Gamma)$  is defined in this frame. A *stationary state (SS)* is a state with a time-independent PDF  $\frac{\partial f}{\partial t} = 0$ , which also implies  $\dot{Z} = \dot{Y} = 0$ . It can be stationary only in a particular rotating frame, so it is characterized by its frame frequency  $\Omega$  and mean fields  $Z, Y$ . We call SSs with  $\Omega = 0$  as *natural states (NS)*, while SSs with  $\Omega \neq 0$  are *traveling wave (TW)* states. For later convenience we define

$$G_{\Omega}^{\pm}(\Gamma) \equiv G(\pm\omega + \Omega, k, q, \beta, \gamma),$$

$$L(x, \Delta) \equiv \frac{\Delta/\pi}{x^2 + \Delta^2}, \quad P(\phi, \nu) \equiv \frac{\nu\sqrt{1+\nu^2}/2\pi}{\nu^2 + \sin^2(\phi/2)}. \quad (7)$$

Note that  $P(\phi, 0) = \delta(\phi)$  and  $P(\phi, \infty) = 1/2\pi$ .

Additionally, we introduce

$$I(\omega, k) \equiv \int e^{i\beta} G(\Gamma) d\Gamma, \quad I_{\Omega}^{\pm} \equiv I(\pm\omega + \Omega),$$

$$J(\omega, k) \equiv \int q e^{i(\beta+\gamma)} G(\Gamma) d\Gamma, \quad J_{\Omega}^{\pm} \equiv J(\pm\omega + \Omega). \quad (8)$$

As will be seen,  $I$  and  $J$  represent some kind of an “effective” complex distribution of  $\omega, k$  for the determination of  $Z$  and  $Y$ , respectively.

*Stationary states.* We will treat SSs in frames where they are stationary, so that the parameter distribution becomes  $G(\Gamma) \rightarrow G_{\Omega}^+(\Gamma)$ . By definition and (3), SSs satisfy  $\frac{\partial \alpha}{\partial t} = 0$ . Substituting this into (4) and taking account of the OA validity condition  $|\alpha| \leq 1$  [20], one finds that all SSs in their own rotating frames are described by

$$\alpha_s e^{i(\Phi+\beta)} = \begin{cases} \frac{\sqrt{k^2 W^2 - \omega^2} - i\omega}{kW} & \text{if } |\omega| \leq |k|W, \\ -i \frac{\omega - \text{sign}(\omega)\sqrt{\omega^2 - k^2 W^2}}{kW} & \text{if } |\omega| > |k|W. \end{cases} \quad (9)$$

The other solution corresponds to an unstable position on the phase circle, and so is never realized.

Using (9) in (3), one can recover the PDF of an SS

$$\frac{f_s(\theta, \Gamma, t)}{G_{\Omega}^+(\Gamma)} = \begin{cases} \delta(\theta - \beta - \arcsin(\frac{\omega}{|k|W}) - \Phi + \pi H(-k)) & \text{if } |\omega| \leq |k|W, \\ \frac{\sqrt{\omega^2 - k^2 W^2}/2\pi}{|\omega - kW \sin(\theta - \beta - \Phi)|} & \text{if } |\omega| > |k|W, \end{cases} \quad (10)$$

with  $H(\cdot)$  denoting a Heaviside function.

*Self-consistency and stability conditions.* Using (9), the self-consistency conditions (SCCs) on  $Y$  (6) become

$$\tilde{F} \equiv \int \frac{dk}{kW} \left\{ \int_{-|k|W}^{|k|W} \sqrt{k^2 W^2 - \omega^2} H_{\Omega}^+ d\Gamma + i \int \omega J_{\Omega}^+ d\Gamma - i \int_{|k|W}^{\infty} \sqrt{\omega^2 - k^2 W^2} [J_{\Omega}^+ - J_{\Omega}^-] d\Gamma \right\} = W. \quad (11)$$

Taking its real and imaginary parts yields two equations, from which the SS parameters  $W, \Omega$  can be determined. They can then be used in (9), (5) to determine  $Z$  as well.

Next, applying the standard procedure [23], i.e. performing a linear stability analysis of (3) above incoherence ( $\alpha = 0$ ) and using a self-consistency argument (see

[9] and references therein), one can show that incoherence loses stability when there exists a solution  $\tilde{\Omega}$  to

$$\begin{cases} \pi \text{Im} \int k J(\tilde{\Omega}, k) dk + \text{Re} \int \frac{k J(\omega + \tilde{\Omega}, k)}{\omega} d\omega dk = 0, \\ \pi \text{Re} \int k J(\tilde{\Omega}, k) dk - \text{Im} \int \frac{k J(\omega + \tilde{\Omega}, k)}{\omega} d\omega dk = 2. \end{cases} \quad (12)$$

Additionally, we can generalize the empirical stability conditions (ESCs) [9] to be [24]

$$\begin{cases} W(\partial_{\Omega} \text{Im} \tilde{F}) + \partial_W \text{Re} \tilde{F} - 1 < 0, \\ (\partial_W \text{Re} \tilde{F} - 1)(\partial_{\Omega} \text{Im} \tilde{F}) - (\partial_W \text{Im} \tilde{F})(\partial_{\Omega} \text{Re} \tilde{F}) > 0, \end{cases} \quad (13)$$

where  $\partial_{W, \Omega}$  are partial derivatives with respect to  $W, \Omega$ , and  $\text{Re}$  ( $\text{Im}$ ) denote real (imaginary) parts. The ESCs (13) give approximate stability conditions for SSs with  $W > 0$ . Although empirical, they work well in most cases but not all [25], so should be used carefully.

*Uncoupled distributions.* In what follows, we assume that the distribution of  $q, \beta, \gamma$  is uncorrelated with  $\omega, k$ :

$$G(\omega, k, q, \beta, \gamma) = g(\omega, k) h(q, \beta, \gamma), \quad g_{\Omega}^{\pm} \equiv g(\pm\omega + \Omega, k), \quad (14)$$

so that the effective  $\omega, k$  distributions (8) become

$$I(\omega, k) = |I| e^{i\phi_I} g(\omega, k), \quad J(\omega, k) = |J| e^{i\phi_J} g(\omega, k), \quad (15)$$

considerably simplifying all formulæ.

Thus, SCCs (11) reduce to

$$F_W(W, \Omega) = W \cos \phi_J / |J|, \quad F_{\Omega}(W, \Omega) = -W \sin \phi_J / |J|, \quad (16)$$

where

$$F_W(W, \Omega) \equiv \int \frac{dk}{kW} \int_{-|k|W}^{|k|W} g_{\Omega}^+ \sqrt{K^2 R^2 - \omega^2} d\omega,$$

$$F_{\Omega}(W, \Omega) \equiv \int \frac{dk}{kW} \left\{ \int \omega g_{\Omega}^+ d\omega - \int_{|k|W}^{+\infty} [g_{\Omega}^+ - g_{\Omega}^-] \sqrt{\omega^2 - k^2 W^2} d\omega \right\} \quad (17)$$

are the full analogues of  $F_{R, \Omega}$  in [9].

The incoherence stability threshold condition (12) becomes

$$\begin{cases} \int \frac{k g(\omega + \tilde{\Omega}, k)}{\omega} d\omega dk = -\frac{2 \sin \phi_J}{|J|}, \\ \int k g(\tilde{\Omega}, k) dk = \frac{2 \cos \phi_J}{\pi |J|}, \end{cases} \quad (18)$$

while the ESCs (13) can be simplified using  $\tilde{F} = |J| e^{i\phi_J} (F_W + i F_{\Omega})$ . Note that, for uncorrelated  $g(\omega, k) = g(\omega) v(k)$ , incoherence stability is fully determined by  $\langle k \rangle \equiv \int k v(k) dk$  and not by the particular form of  $v(k)$ , as noticed previously for  $q_i = 1, \beta_i = \gamma_i = 0$  [9, 26].

Furthermore, from (5),(6) and (9) it follows that

$$R = |I|W/|J|, \quad \Psi - \Phi = \phi_I - \phi_J, \quad (19)$$

*System reduction.* It is evident that all macroscopic properties of the SSs are completely characterized by  $g(\omega, k)$ ,  $|J|$  and  $\phi_J$ , while  $|I|, \phi_I$  serve merely to specify  $Z$  (19). Thus the SSs of systems with different  $h(q, \beta, \gamma)$  but the same  $|J|, \phi_J$  have same parameters except that  $Z$  is rescaled and phase-shifted. Therefore, one can obtain the parameters of the SSs for (1) as

$$\Omega = \Omega_{SK}, \quad W = |J|R_{SK}, \quad R = |I|R_{SK}, \quad (20)$$

where  $SK$  denotes parameters of a reduced model

$$\dot{\theta}_i = \omega_i - \frac{k_i|J|}{N} \sum_{j=1}^N \sin(\theta_i - \theta_j - \phi_J) \quad (21)$$

with the same distribution of  $\omega_i, k_i$  defined by  $g(\omega, k)$ . Although rather hard to prove rigorously (apart from incoherence, for which it follows from (18)), the stability of the corresponding states of (21) will also be the same, as indicated by ESCs (13) and numerical evidence.

Hence, for *any* distribution of  $q, \beta, \gamma$  obeying (14), one can reduce consideration of (1) to the much simpler, equivalent Sakaguchi-Kuramoto model (21). This elegant and unexpected result is illustrated in Fig. 1. Such reduction, however, concerns only SSs ( $t \rightarrow \infty$ ), whereas full evolutions  $Z(t), Y(t)$  cannot be obtained from (20); the microscopic structures of the “equivalent” SSs might also be different. Note that, from (20),  $W \leq |J|, R \leq |I|$  i.e. the distribution of  $q, \beta, \gamma$  imposes an upper bound on the SSs’ mean field strengths, which they cannot exceed however strong the coupling is.

*Glassy states.* An oscillator glass is a state where the distribution of phases  $\theta_i(t)$  is always uniform and indistinguishable from incoherence but, in contrast to the latter, the oscillators adjust their frequencies and move synchronously. Relative to the natural frame, the uniform phase distribution of incoherence is achieved through the phases moving asynchronously (dynamic disorder), whereas in the glassy state the phases are frozen in random positions (static disorder). Glassy states are characterized by vanishing  $R$  and nonvanishing  $W$ :

$$R = 0, \quad W > 0 \Leftrightarrow |I| = 0, \quad |J| > 0. \quad (22)$$

Representing  $h(q, \beta, \gamma) \equiv h_1(\beta)h_2(q, \gamma|\beta)$ , with a  $\beta$ -conditional distribution  $h_2(q, \gamma|\beta)$ , (22) takes form

$$\int e^{i\beta} h_1(\beta) d\beta = 0, \quad \int q e^{i(\beta+\gamma)} h_1(\beta) h_2(q, \gamma|\beta) dq d\beta d\gamma \neq 0. \quad (23)$$

The first of (23) can be satisfied, e.g. if the  $h_1(\beta)$  have equal peaks at opposite  $\beta$  (i.e. separated by  $\pi$ ):

$$h_1(\beta) \sim \sum_n a_n [P(\beta - \phi_n, \nu_n) + P(\beta - \phi_n - \pi, \nu_n)], \quad \forall a_n, \nu_n, \phi_n \quad (24)$$

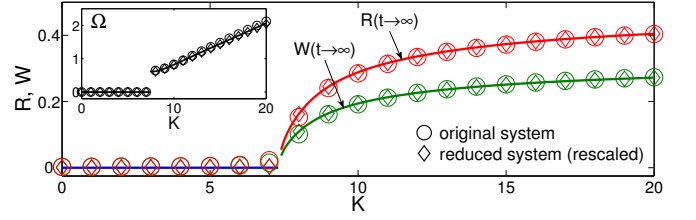


FIG. 1. (Color online) Dependences of the SS parameters  $W$  (green),  $R$  (red) and  $\Omega$  (black) on coupling  $K$ , as calculated for the original system (1) (circles) and obtained from the reduced system (21),(20) (diamonds); solid lines show theoretical predictions. The distribution for the original system was:  $g(\omega, k) \sim \delta(k - K)[\omega^2 + e^{-\omega^2}]^{-1}$ ,  $h(q, \beta, \gamma) \sim e^{-(q-1)^2/0.02}[P(\beta - \pi/8, 0.1) + P(\beta + \pi/2, 0.05)]P(\gamma - \pi/3, 0.2)$ . The simulations used  $N = 10^5$  oscillators and a Runge-Kutta 6th order method with time-step 0.01 s for 500 s; the values are averages over the last 100 s.

For the second condition, one requires the distribution of  $q, \gamma$  to be specifically correlated with  $\beta$ . The simplest choices are  $\gamma_i = -\beta_i + \phi_0$  ( $h_2(q, \gamma|\beta) \sim \delta(\gamma + \beta - \phi_0)$ ), or  $q_i = \cos(\beta_i + \phi_0)$ , but there are many others (e.g.  $h_2(q, \gamma|\beta) \sim L(q - \cos \beta, \Delta_q)$ ).

However, by definition, the glassy state should have a uniform marginal phase distribution

$$\rho(\theta, t) \equiv \int f(\theta, \Gamma, t) d\Gamma = 1/2\pi. \quad (25)$$

As seen from (10), this is satisfied when  $h_1(\beta) = 1/2\pi$  (corresponds to (24) with  $\nu_n \rightarrow \infty$ ). Thus, only for uniform  $\beta$  and suitably chosen  $h_2(q, \gamma|\beta)$  are true glassy states possible; states satisfying (22) but not (25) will be called *pseudoglassy*. Note, that the glassy and pseudoglassy states are essentially SSs with particular microscopic structures, and that they can have  $\Omega \neq 0$ .

As a simplified picture, the distinctions between the states can be understood in terms of a group of people doing cyclical exercises, each with their own tempo and other parameters. The incoherent state is when everyone proceeds independently; synchronized is when they all move synchronously, at any given time having identical poses; pseudoglassy is when they all exercise at the same tempo but remain pairwise in opposite poses; and glassy is when they synchronize their tempos, but always remain in the independent random poses, thus representing a kind of synchronous disorder. Figure 2 illustrates these four different states in the natural frame, where the population does not move as a whole ( $\langle \omega \rangle = 0$ ). In the  $\theta, \omega$ -plane the only difference between incoherence and the glassy state is that, in the latter, phases within the glassy cluster ( $|\omega| < |k|W$ ) do not move, remaining frozen around random angles  $\beta$ , as seen in  $\theta, \beta$  plane.

*Relaxation to an SS.* There has been a continuing controversy about the relaxation of  $R$  for glassy states. Some authors reported it to be algebraic [10, 13], others exponential [11, 12, 14]. Although one should not expect it

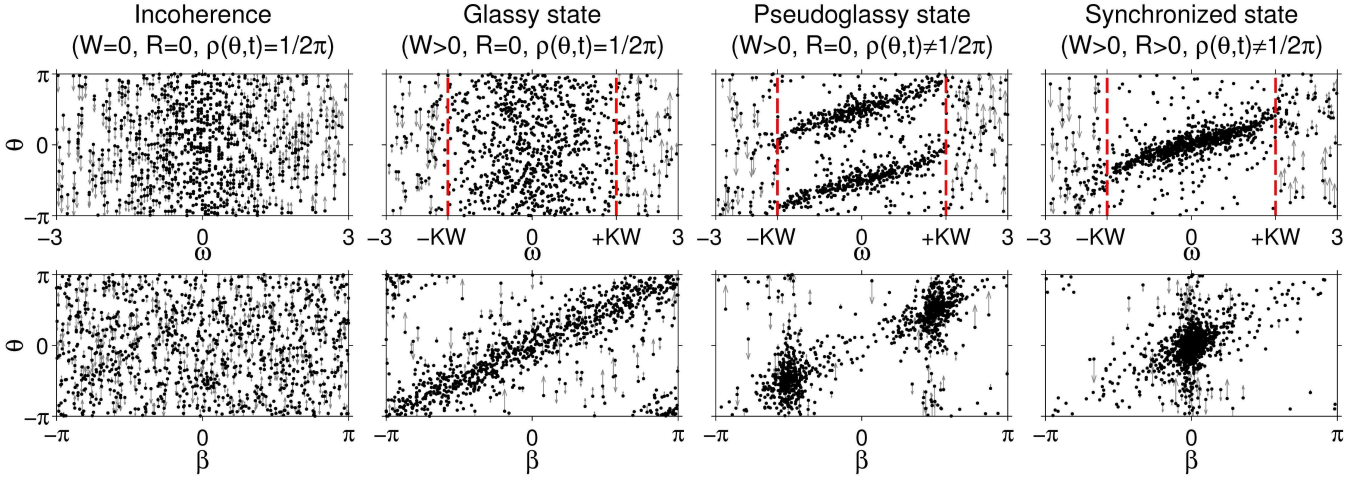


FIG. 2. (Color online) The different kinds of states, as illustrated by snapshots of positions in the  $\theta, \omega$  (upper panel) and  $\theta, \beta$  (bottom panel) planes of (the same) 1000 randomly selected oscillators (out of  $N = 25600$ ). Tiny vertical arrows show the movements of the oscillators during 0.25 s. Red dashed lines show boundaries between the clusters and incoherent populations. In all cases  $g(\omega, k) = L(\omega, 1)\delta(k - K)$ ,  $h(q, \beta, \gamma) = h_1(\beta)\delta(q - 1)\delta(\beta + \gamma)$  and (a,b)  $K = 1$ ,  $h_1(\beta) = 1/2\pi$ ; (c,d)  $K = 3$ ,  $h_1(\beta) = 1/2\pi$ ; (e,f)  $K = 3$ ,  $h_1(\beta) = (1/2)[P(\beta - \pi/2, 0.1) + P(\beta + \pi/2, 0.1)]$ ; (g,h)  $K = 3$ ,  $h_1(\beta) = P(\beta, 0.1)$ . The pictures refer to the natural frame ( $\langle \omega \rangle = 0$ ), and all states have  $\Omega = 0$ ; for TWs ( $\Omega \neq 0$ ) one would observe similar pictures in the rotating frames (or add  $\Omega$  to the “speed” of all oscillators in the natural frame). A dynamical version of this figure is available [27].

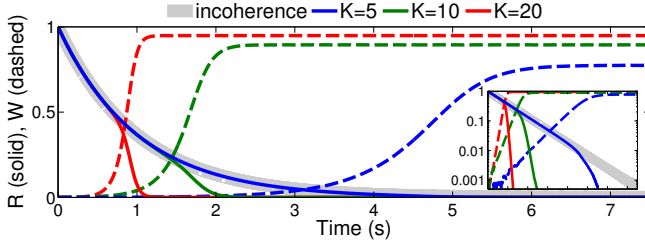


FIG. 3. (Color online) Time-evolution of  $W(t), R(t)$  from the initial conditions  $R(0) = 1$  ( $\theta_i = 0$ ) for the glassy state ( $g(\omega, k) = \delta(k - K)L(\omega, 1)$ ,  $h(q, \beta, \gamma) = \delta(q - 1)\delta(\beta + \gamma)/2\pi$ ) for different constant couplings  $k_i = K$ . The thick gray line on the background shows the relaxation of  $R(t)$  in the case of incoherence (which is independent from  $K$ ). The inset shows the results for a logarithmic ordinate scale. The simulations used  $N = 10^6$  oscillators and a Runge-Kutta 6th order method with time-step 0.01 s.

to be the same for different KM variants, we find that, for (1), relaxation is more complex, though close to exponential at short and long times, as shown in Fig. 3.

Surprisingly, for  $\beta$ -distributions (24) the relaxation of  $R$  to incoherence is fully universal: it does not depend on coupling strength,  $\{a_n, \nu_n, \phi_n\}$  in (24) or even on the form of  $h_2(q, \gamma|\beta)$ . Furthermore, the relaxation to glassy and pseudoglassy states occurs in two stages. The phases first begin to disorder in the same way as for incoherence, so that  $R$  decays exponentially with a fixed, coupling-independent exponent. At the same time, oscillators are also slowly entrained while passing their equilibrium positions. When the effective field of the entrained oscillators, characterized by  $W$ , becomes large enough, they

begin to force unentrained ones to take their positions, so the relaxation switches to a faster, coupling-dependent exponent and  $R$  then decays more rapidly. This “switch” occurs sooner for stronger coupling.

*Conclusions.* We have generalized the approach of [9] to a wider case (1) so that, using (11), (12) and (13), one can immediately obtain the macroscopic parameters of possible stationary states [28]. In addition to the particular cases included within the framework of [9], the KM (1) trivially includes the Sakaguchi-Kuramoto model [29], as well as more sophisticated cases such as the KM with shear diversity [26, 30, 31], allowing one readily to reproduce and extend many previous results [32]. Remarkably, the behavior of (1) with any distribution  $h(q, \beta, \gamma)$  (14) in the asymptotic limit  $t \rightarrow \infty$  can be obtained from the simple Sakaguchi-Kuramoto model (21),(20).

However, our main finding is that the model (1) can exhibit an oscillator glass state, which has very exotic properties and which appears when both (22) and (25) are satisfied. This discovery opens new horizons for KM-related investigations, e.g. it will be interesting to see whether glassy states can appear when additional complications [33–39] like time lags are introduced into (1). Finally, it should now be possible to create, observe and study oscillator glass in real systems, where a variety of novel phenomena may be anticipated. For example, if some physical quantity can be associated with the weighted mean field  $Y$  (2) in laser arrays obeying the KM [5], it might be possible to construct a laser with zero intensity ( $R = 0$ ) but other nonvanishing effects ( $W > 0$ ). Similar considerations apply to other KM applications.

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- [21] Normalization  $\sum q_i = 1$  is inapplicable when  $q_i = \cos(\gamma_i)$ ;  $\sum |q_i| = 1$ , on the other hand, fails for a Lorenzian distribution of  $q$ . A very useful normalization is  $\int |J(\omega, k)| d\omega dk = 1$ , where  $J(\omega, k)$  is defined in (8): it is always applicable and assures  $W \leq 1$ . However, when  $q = 1$  and  $\beta$  is distributed (so  $Y = Z$ ), it will lead to a rescaling of  $q$  (so  $Y \neq Z$ ), which might be inconvenient.
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- [25] For example, ESCs may predict parts of the stable TWs to be unstable for multimodal-delta distributions of  $K$  and some  $\phi_J \neq 0$ . They also can fail in the presence of standing waves [9], but seem to be exact for  $\phi_J = 0$  and unimodal over  $\omega$  distributions  $G(\Gamma)$ .
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- [28] Note that the approach might be not valid when the OA ansatz fails [18, 19], e.g. for distributions that are discontinuous over  $\omega$ , like  $G(\Gamma) \sim \sum_n a_n \delta(\omega - \omega_n)$ .
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- [32] For example, in [26] the authors considered the model  $\theta_i = \tilde{\omega}_i - \frac{K(1+s_i c)}{N \cos \beta_i} [\sum_i \sin(\theta_i - \theta_j - \beta_i) + N \sin \beta_i]$  with  $\tan \beta_i = (c - s_i)/(1 + s_i c)$  and  $\tilde{\omega}_i, s_i$  distributed according to particular  $\tilde{g}(\tilde{\omega}, s)$ . But this is in fact (1) with  $\omega_i = \tilde{\omega}_i + K(s_i - c), k_i = K(1 + s_i c)/\cos \beta_i, q_i = 1, \gamma_i = 0$  (so  $Y = Z$ ). Thus, the distribution will be  $G(\Gamma) = \delta(q-1)\delta(\gamma) \int \delta(k - \frac{K(1+sc)}{\cos \beta}) \delta(\beta + \arctan \frac{s-c}{1+sc}) \tilde{g}(\omega - K(s-c), s) ds$ , which then can be substituted into (11), (12), (13) to reproduce and extend many results of [26] (after integration over  $k, \beta$  all becomes quite simple).
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